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LETTER TO THE EDITOR

A new algebraic Bethe ansatz for *gl*(2, 1) invariant vertex models

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Abstract. The algebraic Bethe ansatz for the integrable vertex model constructed from the four-dimensional $[b, \frac{1}{2}]$ representation of the superalgebra $gl(2,1)$ is calculated using a ferromagnetic reference state. This Bethe ansatz was known only for the three-dimensional $[\frac{1}{2}]_+$ representation leading to the supersymmetric t-J model. The necessary modification of the nested algebraic Bethe ansatz scheme and generalizations to related models are discussed.

Integrable vertex models built from low-dimensional representations of the superalgebra $gl(2,1)$ have attracted considerable interest, because they allow one to construct integrable models of interacting electrons in one spatial dimension. The most prominent example is the supersymmetric t-J model [1–3], which is obtained from the transfer matrix of the vertex model based on the three-dimensional fundamental $[\frac{1}{2}]_+$ representation [4, 5].

The transfer matrix for vertex models built from the one parametric four-dimensional $[b, \frac{1}{2}]$ representation leads to a model of interacting electrons, where the interaction strength is determined by the free parameter $|b| > \frac{1}{2}$ [6]. This model has been solved by means of the coordinate Bethe ansatz [7–9].

The nested algebraic Bethe ansatz for the supersymmetric t-J model can start with either the empty lattice or the fully polarized ferromagnetic state as a pseudovacuum to construct the Bethe vectors from. Considering different possibilities to solve the nesting, there are three sets of Bethe ansätze [4]. For the model built from the four-dimensional $[b, \frac{1}{2}]$ representations, only two Bethe ansätze have been reported [10–12]. They start either with the empty or the completely filled band as pseudovacuum. In addition, these two possibilities are related by the automorphism $b \to -b$ of the algebra $gl(2,1)$ [11].

In this letter we show that, in contrast to earlier assumptions, it is indeed possible to find a third Bethe ansatz for this model starting from a ferromagnetic pseudovacuum§ where every lattice site is occupied by a single electron with spin $\uparrow$. It is, however, necessary to modify the usual scheme of the nested algebraic Bethe ansatz. The resulting Bethe equations show an interesting symmetry between the spectral parameters of the first and the second level, which has not been observed before in an electronic system.

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§ Our notation refers to the electronic model. In [12] the authors choose the grading of the $|b+\frac{1}{2}, 0, 0\rangle$ and the $|b-\frac{1}{2}, 0, 0\rangle$ state as fermionic and call the corresponding pseudovacuum ferromagnetic. In the electronic model these states have to be identified with an empty and doubly occupied lattice site, respectively, and thus are clearly not ferromagnetic.

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Before presenting our new results we have to fix some notation. The basis vectors of the irreducible representations of $gl(2,1)$ can be labelled by the $u(1)$ charge B, the total spin S and the z-component of the spin S^z: $|B,S,S^z\rangle$, [13, 14]. For the three-dimensional $[\frac{1}{2}]_+$ representation we choose the basis $|1,0,0\rangle$, $|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle$, $|\frac{1}{2},\frac{1}{2},+\frac{1}{2}\rangle$. In the t-J model these vectors are identified with an empty site and a single electron with spin $\downarrow$ or $\uparrow$. The first state is considered as bosonic (grading 0), the latter two as fermionic (grading 1). A basis of the four-dimensional $[b,\frac{1}{2}]$ representation is given by the vectors $|b,\frac{1}{2},+\frac{1}{2}\rangle$, $|b,\frac{1}{2},-\frac{1}{2}\rangle$, $|b-\frac{1}{2},0,0\rangle$, $|b+\frac{1}{2},0,0\rangle$. They correspond to the electronic states with a single electron with spin $\uparrow$ or $\downarrow$, a doubly occupied site and an empty site. The grading is $[1,1,0,0]$.

On the tensor product of two $[\frac{1}{2}]_+$ representations the Yang–Baxter equation is solved by the R-matrix r^{33}:

$$r^{33}(\lambda) = a(\lambda)\mathrm{id}_9 + b(\lambda)\Pi_{33}. \tag{1}$$

Here id_9 denotes the 9×9 identity matrix and Π_{33} is the graded permutation operator with matrix elements $(\Pi_{33})^{i_1,j_1}_{i_2,j_2} = (-1)^{[i_1][i_2]}\delta_{i_1,j_2}\delta_{i_2,j_1}$, where $[x]$ denotes the grading of an object x. The functions a and b are given by $a(\lambda)=\lambda/(\lambda+1)$ and $b(\lambda)=1/(\lambda+1)$.

On the tensor product $[\frac{1}{2}]_+\otimes[b,\frac{1}{2}]$ the R-matrix R can be represented in terms of $gl(2,1)$ operators. Their matrix representations can be found in the literature [13, 14]:

$$\begin{aligned} R(\lambda) = a(\lambda)\mathrm{id}_{12} + b(\lambda)[&-2\hat{Q}^z_3\otimes\hat{Q}^z_4 - \hat{Q}^+_3\otimes\hat{Q}^-_4 - \hat{Q}^-_3\otimes\hat{Q}^+_4 + 2\hat{B}_3\otimes\hat{B}_4 \\ &+2\hat{W}^-_3\otimes\hat{V}^+_4 - 2\hat{W}^+_3\otimes\hat{V}^-_4 + 2\hat{V}^-_3\otimes\hat{W}^+_4 - 2\hat{V}^+_3\otimes\hat{W}^-_4]. \end{aligned} \tag{2}$$

We used the subscripts 3 and 4 to indicate which operators act in the three-dimensional $[\frac{1}{2}]_+$ representation and which in the four-dimensional $[b,\frac{1}{2}]$ representation. At this point we note that the construction of an R-matrix such as (2) is possible with any irreducible representation of $gl(2,1)$ instead of the $[b,\frac{1}{2}]$ representation. This R-matrix is a solution of the Yang–Baxter equation on the tensor product $[\frac{1}{2}]_+\otimes[\frac{1}{2}]_+\otimes[b,\frac{1}{2}]$:

$$r^{33}_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)r^{33}_{12}(\lambda-\mu). \tag{3}$$

(Subscripts denote the spaces in which the R-matrix acts.) From the R-matrix (2) we construct the monodromy matrix by taking matrix products in the first component of the tensor product—the auxiliary or matrix space—and tensor products of the second components—the quantum spaces:

$$T(\mu)^{a,\alpha_1,\ldots,\alpha_L}_{b,\beta_1,\ldots,\beta_L} = R(\mu)^{a,a_L}_{\alpha_L,\beta_L}R(\mu)^{a_L,a_{L-1}}_{\alpha_{L-1},\beta_{L-1}}\ldots R(\mu)^{a_3,a_2}_{\alpha_2,\beta_2}R(\mu)^{a_2,b}_{\alpha_1,\beta_1}(-1)^{\sum_{i=2}^{L}([\alpha_i]+[\beta_i])\sum_{j=1}^{i-1}[\alpha_j]}. \tag{4}$$

The additional signs are a consequence of the multiplication rule in graded tensorproducts, which reads $(A\otimes B)(v\otimes w) = (-1)^{[B][v]}(Av)\otimes(Bw)$. As a consequence of equation (3) the monodromy matrix satisfies the following Yang–Baxter equation:

$$r^{33}_{12}(\lambda-\mu)T_{13}(\lambda)T_{23}(\mu) = T_{23}(\mu)T_{13}(\lambda)r^{33}_{12}(\lambda-\mu). \tag{5}$$

Here space 3 is the L-fold tensorproduct of $[b,\frac{1}{2}]$ representations. From the monodromy matrix the transfer matrix is obtained by taking the supertrace in the auxiliary space: $\tau(\mu) = \sum_a(-1)^{[a]}T(\mu)^a_a$. The Yang–Baxter equation (5) implies that transfer matrices with different spectral parameters commute.

To diagonalize the transfer matrix τ by means of the algebraic Bethe ansatz, we have to find a suitable reference state to construct the Bethe vectors from. Instead of specifying this state, as is done usually, we consider a subspace V of the L-fold tensor product of the $[b,\frac{1}{2}]$ representation spaces. V is spanned by the 2^L vectors

$$|\alpha_1,\ldots,\alpha_L\rangle = |\alpha_1\rangle_1\otimes\cdots\otimes|\alpha_L\rangle_L \qquad \alpha_i = 1,2. \tag{6}$$

Here we denote by $|\alpha\rangle$ the vectors $|1\rangle = |b, \frac{1}{2}, \frac{1}{2}\rangle$ (electron with spin $\uparrow$) and $|2\rangle = |b - \frac{1}{2}, 0, 0\rangle$ (two electrons). From the representation (2) of the R-matrix we can determine the action of the monodromy matrix on an arbitrary vector $|v\rangle$ out of V:

$$T(\mu)|v\rangle = \begin{pmatrix} A_{1,1}(\mu) & A_{1,2}(\mu) & 0 \\ A_{2,1}(\mu) & A_{2,2}(\mu) & 0 \\ C_1(\mu) & C_2(\mu) & D(\mu) \end{pmatrix} |v\rangle. \tag{7}$$

The subspace V has several peculiar properties. It is an eigenspace of the operator $D(\mu)$ corresponding to the eigenvalue $((\mu + b - \frac{1}{2})/(\mu+1))^L$. Furthermore, the operators $A_{i,j}(\mu)$ map V back onto V. For later convenience we define their restrictions onto V according to $\tilde{A}(\mu)_{i,j} = P_V A_{i,j}(\mu) P_V$, where P_V is the projector onto V. In (7) the operators $A_{i,j}$ can be replaced by $\tilde{A}_{i,j}$ without problem. Finally, the matrix $\tilde{A}$ itself is solution of a Yang–Baxter equation:

$$r^{(1)}_{12}(\lambda - \mu)\tilde{A}_{13}(\lambda)\tilde{A}_{23}(\mu) = \tilde{A}_{23}(\mu)\tilde{A}_{13}(\lambda)r^{(1)}_{12}(\lambda - \mu). \tag{8}$$

Here $r^{(1)}(\mu)$ is the 4×4 R-matrix of a model with one bosonic and one fermionic state:

$$r^{(1)}(\mu) = a(\mu)\mathrm{id}_4 + b(\mu)\Pi_{BF}. \tag{9}$$

For this R-matrix we have the Yang–Baxter equation:

$$r^{(1)}_{12}(\lambda - \mu)r^{(1)}_{13}(\lambda)r^{(1)}_{23}(\mu) = r^{(1)}_{23}(\mu)r^{(1)}_{13}(\lambda)r^{(1)}_{12}(\lambda - \mu). \tag{10}$$

We now make the following ansatz for the eigenvectors of the transfer matrix $\tau(\mu) = A_{11}(\mu) - A_{22}(\mu) - D(\mu)$:

$$|\lambda_1, \ldots, \lambda_n\rangle = F^{a_1,\ldots,a_n,\alpha_1,\ldots,\alpha_L} C_{a_1}(\lambda_1) \ldots C_{a_n}(\lambda_n)|\alpha_1, \ldots, \alpha_L\rangle. \tag{11}$$

This ansatz differs from the usual Bethe ansatz because the reference state, on which the creation operators C_i act, remains indeterminate. We still have to calculate the n spectral parameters λ_i and the $2^n \times 2^L$ amplitudes F. An ansatz of this type was first proposed in [15] where vertex models combining different representation of $su(3)$ are considered.

We can now proceed following the usual steps of a nested algebraic Bethe ansatz. From the Yang–Baxter equation (3) we derive commutation relations for the operator valued entries of the monodromy matrix. This allows the calculation of the action of the diagonal parts on a Bethe vector (11). We find that a vector (11) is an eigenvector of the transfer matrix τ, if F is an eigenvector of the second transfer matrix $\tau^{(1)}$ and if the spectral parameters λ_k, $(k = 1, \ldots, n)$, are solutions of the equations

$$\left(\frac{\lambda_k + b - s}{\lambda_k + 1}\right)^L F^{b_1,\ldots,b_n,\beta_1,\ldots,\beta_L} = \tau^{(1)}(\lambda_k)^{b_1,\ldots,b_n,\beta_1,\ldots,\beta_L}_{a_1,\ldots,a_n,\alpha_1,\ldots,\alpha_L} F^{a_1,\ldots,a_n,\alpha_1,\ldots,\alpha_L}. \tag{12}$$

Here the nested transfer matrix $\tau^{(1)}$ corresponds to an inhomogeneous vertex model and is defined as

$$\begin{aligned} \tau^{(1)}(\mu)^{b_1,\ldots,b_n,\beta_1,\ldots,\beta_L}_{a_1,\ldots,a_n,\alpha_1,\ldots,\alpha_L} &= \mathrm{str}(T^{(1)}(\mu)) \\ &= (-1)^{[c]}\tilde{A}_{c,c_n}(\mu)^{\beta_1,\ldots,\beta_L}_{\alpha_1,\ldots,\alpha_L} r^{(1)}(\mu - \lambda_n)^{c_n,c_{n-1}}_{b_n,a_n} \ldots r^{(1)}(\mu - \lambda_1)^{c_1,c}_{b_1,a_1} \\ &\quad \times (-1)^{\sum_{i=1}^{n-1}([c]+[c_i])([b_i]+1)+([c]+[c_n])[b_n]}. \end{aligned} \tag{13}$$

This transfer matrix still contains a part of the first transfer matrix τ, namely the operator $\tilde{A}_{c,c_n}(\mu)$. The monodromy matrix $T^{(1)}$ corresponds to a vertex model with a two-dimensional auxiliary space and $n + L$ two-dimensional quantum spaces. The quantum spaces at sites $n + 1, \ldots, L + n$ correspond to the subspace V. The signs imply that the grading of the quantum spaces at sites $1, \ldots, n-1$ has been changed from $[0,1]$ to $[1,0]$. The reason for this change is the fact that the operator C_1 is fermionic and C_2 is bosonic.

Because $r^{(1)}$ and $\tilde{A}$ are intertwined by the same R-matrix (cf (8) and (10)) we have a Yang–Baxter equation for $T^{(1)}$ as well:

$$r_{12}^{(1)}(\lambda-\mu)T_{13}^{(1)}(\lambda)T_{23}^{(1)}(\mu) = T_{23}^{(1)}(\mu)T_{13}^{(1)}(\lambda)r_{12}^{(1)}(\lambda-\mu). \tag{14}$$

The diagonalization of the transfer matrix $\tau^{(1)}$ by a second Bethe ansatz is standard [16]. Choosing the following pseudovacuum

$$|0\rangle^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}_1 \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix}_n \otimes |1, \ldots, 1\rangle \tag{15}$$

the eigenstates of $\tau^{(1)}$ are parametrized by rapidities ν_j satisfying

$$\left(\frac{\nu_j + 2b}{\nu_j + b + \frac{1}{2}}\right)^L = \prod_{i=1}^{n} a(\nu_j - \lambda_i) \qquad j = 1, \ldots, n_1. \tag{16}$$

The corresponding eigenvalue is

$$\Lambda^{(1)}(\mu) = \prod_{j=1}^{n_1} \frac{1}{a(\nu_j - \mu)} \left[\left(\frac{\mu + 2b}{\mu + 1}\right)^L - \prod_{i=1}^{n} a(\mu - \lambda_i)\left(\frac{\mu + b + \frac{1}{2}}{\mu + 1}\right)^L\right]. \tag{17}$$

Note that in (15) the vector $|1, \ldots, 1\rangle$ from the subspace V corresponds to the state where every lattice site is occupied by a single spin $\uparrow$ electron. This shows that the Bethe ansatz starts with a ferromagnetic reference state.

Inserting expression (17) into (12) leads to the Bethe equations for the spectral parameters λ_k:

$$\left(\frac{\lambda_k + b - \frac{1}{2}}{\lambda_k + 2b}\right)^L = \prod_{j=1}^{n_1} \frac{1}{a(\nu_j - \lambda_k)} \qquad k = 1, \ldots, n. \tag{18}$$

Finally, the eigenvalues of the transfer matrix τ are

$$\Lambda^{[\frac{1}{2}]_+,[b,\frac{1}{2}]}(\mu) = -\left(\frac{\mu + b - \frac{1}{2}}{\mu + 1}\right)^L \prod_{i=1}^{n} \frac{1}{a(\mu - \lambda_i)} + \prod_{i=1}^{n} \frac{1}{a(\mu - \lambda_i)} \Lambda^{(1)}(\mu). \tag{19}$$

In the electronic model the numbers n and n_1 are related to the number of electrons N_{e} and z-component of the spin S^z of the corresponding Bethe vector according to

$$N_{\mathrm{e}} = L - n + n_1 \qquad S^z = \tfrac{1}{2}(L - n - n_1). \tag{20}$$

From a fusion of two $R^{[\frac{1}{2}]_+,[b,\frac{1}{2}]}$ matrices we can obtain the R-matrix with a four-dimensional $[\frac{3}{2}, \frac{1}{2}]$ representation in the auxiliary space [10, 11]. For the eigenvalues of the corresponding transfer matrix we have the relation:

$$\Lambda^{[\frac{1}{2}]_+,[b,\frac{1}{2}]}(\mu + \tfrac{1}{2})\Lambda^{[\frac{1}{2}]_+,[b,\frac{1}{2}]}(\mu - \tfrac{1}{2}) = \Lambda^{[1]_+,[b,\frac{1}{2}]}(\mu) + \Lambda^{[\frac{3}{2},\frac{1}{2}],[b,\frac{1}{2}]}(\mu). \tag{21}$$

In contrast to the Bethe ansätze presented in [10, 11], the different parts of the left-hand side of this equation can be assigned to $\Lambda^{[\frac{3}{2},\frac{1}{2}],[b,\frac{1}{2}]}$ and $\Lambda^{[1]_+,[b,\frac{1}{2}]}$ without ambiguities. This allows to determine the eigenvalues $\Lambda^{[\frac{3}{2},\frac{1}{2}],[b,\frac{1}{2}]}(\mu)$. Using the analytic properties of the eigenvalues together with the Bethe equations we obtain the general expression

$\Lambda^{[b_1,\frac{1}{2}],[b_2,\frac{1}{2}]}(\mu)$ for the eigenvalues of the transfer matrix with a $[b_1, \frac{1}{2}]$ representation in the auxiliary space and a $[b_2, \frac{1}{2}]$ representation in the quantum space:

$$\begin{aligned}\Lambda^{[b_1,\frac{1}{2}],[b_2,\frac{1}{2}]}(\mu) = &-\left(\frac{\mu+\frac{1}{2}(b_1+b_2)+\frac{1}{2}}{\mu-\frac{1}{2}(b_1+b_2)-\frac{1}{2}}\right)^L \prod_{i=1}^{n}\frac{\mu+\mathrm{i}\lambda_i-\frac{1}{2}b_1-\frac{1}{4}}{\mu+\mathrm{i}\lambda_i+\frac{1}{2}b_1+\frac{1}{4}}\prod_{j=1}^{n_1}\frac{\mu+\mathrm{i}\nu_j+\frac{1}{2}b_1-\frac{1}{4}}{\mu+\mathrm{i}\nu_j-\frac{1}{2}b_1+\frac{1}{4}}\\ &-\left(\frac{\mu-\frac{1}{2}b_1+\frac{1}{2}b_2}{\mu-\frac{1}{2}(b_1+b_2)-\frac{1}{2}}\,\frac{\mu+\frac{1}{2}(b_1)-\frac{1}{2}(b_2)}{\mu-\frac{1}{2}(b_1+b_2)+\frac{1}{2}}\right)^L\\ &\times\prod_{i=1}^{n}\frac{\mu+\mathrm{i}\lambda_i-\frac{1}{2}b_1+\frac{3}{4}}{\mu+\mathrm{i}\lambda_i+\frac{1}{2}b_1+\frac{1}{4}}\prod_{j=1}^{n_1}\frac{\mu+\mathrm{i}\nu_j+\frac{1}{2}b_1+\frac{3}{4}}{\mu+\mathrm{i}\nu_j-\frac{1}{2}b_1+\frac{1}{4}}\\ &+\left(\frac{\mu+\frac{1}{2}(b_1+b_2)+\frac{1}{2}}{\mu-\frac{1}{2}(b_1+b_2)-\frac{1}{2}}\,\frac{\mu-\frac{1}{2}b_1+\frac{1}{2}b_2}{\mu-\frac{1}{2}(b_1+b_2)+\frac{1}{2}}\right)^L\\ &\times\prod_{i=1}^{n}\frac{\mu+\mathrm{i}\lambda_i-\frac{1}{2}b_1+\frac{3}{4}}{\mu+\mathrm{i}\lambda_i+\frac{1}{2}b_1+\frac{1}{4}}\prod_{j=1}^{n_1}\frac{\mu+\mathrm{i}\nu_j+\frac{1}{2}b_1-\frac{1}{4}}{\mu+\mathrm{i}\nu_j-\frac{1}{2}b_1+\frac{1}{4}}\\ &+\left(\frac{\mu+\frac{1}{2}b_1-\frac{1}{2}b_2}{\mu-\frac{1}{2}(b_1+b_2)-\frac{1}{2}}\right)^L\prod_{i=1}^{n}\frac{\mu+\mathrm{i}\lambda_i-\frac{1}{2}b_1-\frac{1}{4}}{\mu+\mathrm{i}\lambda_i+\frac{1}{2}b_1+\frac{1}{4}}\prod_{j=1}^{n_1}\frac{\mu+\mathrm{i}\nu_j+\frac{1}{2}b_1+\frac{3}{4}}{\mu+\mathrm{i}\nu_j-\frac{1}{2}b_1+\frac{1}{4}}.\end{aligned} \tag{22}$$

To use the R-matrix R^{44} of [11] we have added an overall factor

$$\left(\frac{\mu+\frac{3}{2}}{\mu+2b+\frac{3}{2}}\,\frac{\mu+\frac{1}{2}}{\mu+2b+\frac{1}{2}}\right)^L$$

and replaced $\mu \to -\mu - \frac{3}{2}b - \frac{1}{4}$, $\lambda_i \to +\mathrm{i}\lambda_i - \frac{3}{2}b + \frac{1}{4}$, $\nu_j \to +\mathrm{i}\nu_j - \frac{3}{2}b - \frac{1}{4}$. The Bethe equations read

$$\left(\frac{\lambda_k+\mathrm{i}(c+1)/2c}{\lambda_k-\mathrm{i}(c+1)/2c}\right)^L = \prod_{j=1}^{n_1}\frac{\lambda_k-\nu_j+\mathrm{i}\frac{1}{2}}{\lambda_k-\nu_j-\mathrm{i}\frac{1}{2}} \qquad k=1,\ldots,n \tag{23}$$

$$\left(\frac{\nu_j-\mathrm{i}/2c}{\nu_j+\mathrm{i}/2c}\right)^L = \prod_{i=1}^{n}\frac{\nu_j-\lambda_i+\mathrm{i}\frac{1}{2}}{\nu_j-\lambda_i-\mathrm{i}\frac{1}{2}} \qquad j=1,\ldots,n_1 \tag{24}$$

where we used the notation $c = 1/(b_2 - \frac{1}{2})$. For the model of electrons with correlated hopping in the notation of [9] this leads to the energies

$$E = -\frac{c+1}{c^2}\sum_{k=1}^{n}\frac{1}{\lambda_k^2+((c+1)/2c)^2} + \frac{1}{c^2}\sum_{j=1}^{n}\frac{1}{\nu_j^2+(1/2c)^2} + 2N_{\mathrm{e}} - 2L. \tag{25}$$

In addition to real solutions λ_k and ν_j of the Bethe equations (23) and (24), we have string solutions of the following structure. For $c > 0$, m spectral parameters $\lambda_{m,j}$ $(j = 1, \ldots, m)$ and $m-1$ spectral parameters $\nu_{m-1,k}$ $(k = 1, \ldots, m-1)$ form one complex string solution:

$$\lambda_{m,j} = \lambda_m + \tfrac{1}{2}\mathrm{i}(n+1-2j) \qquad j=1,\ldots,m \tag{26}$$

$$\nu_{m-1,k} = \lambda_m + \tfrac{1}{2}\mathrm{i}(n-2k) \qquad k=1,\ldots,m-1. \tag{27}$$

Similarly, for $c < 0$ we have strings combining $m-1$ spectral parameters $\lambda_{m-1,j}$ and m spectral parameters $\nu_{m,k}$.

Before closing we would like to add a few comments. Looking at the Bethe equations (23) and (24), the usual distinction between first level and nesting is not obvious. In both we find terms to the power L on the left-hand side. In practice, this fact allows the number n_1 of spectral parameters in the nesting to exceed the number n of spectral parameters in the first level. Physically, this situation is easily understandable. For $n_1 = 0$ there are only n holes in the ferromagnetic reference state, for $n = 0$ we have n_1 spin $\downarrow$ electrons moving in front of the fully polarized background of spin $\uparrow$ electrons. In the limit $b = \frac{1}{2}(c = \infty)$ we recover the Bethe equations for the supersymmetric t-J model derived by Essler and Korepin [4].

It should be noted, that a second Bethe ansatz can be obtained by choosing the pseudo vacuum $|0\rangle^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \otimes \cdots \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_n \otimes |2, \ldots, 2\rangle$ instead of (15) in the nesting. This leads to the known results from [10, 11].

As we have pointed out, the R-matrix (2) can be defined for an arbitrary irreducible representation of $gl(2, 1)$ in the quantum space. The Bethe ansatz corresponding to the transfer matrix with a $[\frac{1}{2}]_+$ representation in the auxiliary space and a $8s$-dimensional $[b, s]$ representation in the quantum space can be calculated along similar lines.

An interesting possibility is the combination of different representations of $gl(2, 1)$ in one transfer matrix. The case of an impurity with a $[b, \frac{1}{2}]$ representation in the t-J model was considered in [17]. There the Bethe equations with the completely filled band of spin $\uparrow$ electrons as reference state were obtained by a particle–hole transformation. These equations can now be rederived by means of the algebraic Bethe ansatz. Remarkably, the impurity becomes visible only in the second set of Bethe equations.

We hope that the method of a modified algebraic Bethe ansatz for supersymmetric $gl(2, 1)$ invariant vertex models presented in this letter is useful for the diagonalization of other systems, where the standard scheme has failed due to the lack of a suitable reference state.

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